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FINITE APERIODIC SEMIGROUPS WITH COMMUTING IDEMPOTENTS AND GENERALIZATIONS

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ABSTRACT

It is proved that any pseudovariety of finite semigroups generated by inverse semigroups, the subgroups of which lie in some proper pseudovariety of groups, does not contain all aperiodic semigroups with commuting idempotents. In contrast we show that every finite semigroup with commuting idempotents divides a semigroup of partial bijections that shares the same subgroups. Finally, we answer in the negative a question of Almeida as to whether a result of Stiffler characterizing the semidirect product of the pseudovarieties of $\mathcal{R}$-trivial semigroups and groups applies to any proper pseudovariety of groups.

1. Introduction

Among the most important and intensively studied classes of semigroups are finite semigroups, regular semigroups and inverse semigroups. Finite semigroups arise as syntactic semigroups of regular languages and as transition semigroups of finite automata. This connection has lead to a large and deep literature on classifying regular languages by means of algebraic properties of their corresponding

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syntactic semigroups. The Eilenberg Variety Theorem [E] establishes a one-one correspondence between so-called varieties of formal languages and pseudovarieties of finite semigroups. A pseudovariety is a collection of finite semigroups closed under homomorphic image, subsemigroups and (finite) direct product. The books by Eilenberg [E], Lallement [L], Pin [P] and Almeida [Al] give many details about this field.

Regular semigroups, that is semigroups S such that for all $s \in S$ there is $t \in S$ such that $sts = s$, have also been intensively studied. Natural examples of regular semigroups include the full transformation semigroup on a set and the semigroup of all matrices over a field. Recently Putcha and Renner have developed a theory of algebraic monoids. In this theory, regular semigroups are naturally associated with reductive algebraic groups. Furthermore, they have developed a notion of "finite monoid of Lie type", a class of finite regular semigroups associated with groups of Lie type. See [Pu] or the survey article [S].

Regular semigroups have also been intensively studied within semigroup theory itself. Among the classes that have received the most treatment is the class of inverse semigroups. These are precisely the regular semigroups whose idempotents commute—that is form a semilattice under multiplication. This property turns out to be equivalent to the fact that every element s in an inverse semigroup S has a unique inverse s^{-1} satisfying $ss^{-1}s = s$ and $s^{-1}ss^{-1} = s^{-1}$. An important example of an inverse semigroup is the semigroup of all partial bijections on a set. This is called the symmetric inverse semigroup and plays the role in inverse semigroup theory that the symmetric group plays in group theory. The Preston–Wagner Theorem is the analogue of the Cayley Theorem and states that every inverse semigroup is faithfully represented by partial bijections. Thus inverse semigroups arise naturally when studying partial automorphisms of a set. They form an important area of study in geometry where they are called pseudogroups of local transformations. See the book [Pe] for background as well as the recent book by Lawson [La].

Historically, these three areas of semigroup theory have developed independently of one another (although for some time results from inverse semigroup theory have been used as models to guide the search for generalizations to regular semigroup theory). In the early 1990's Ash [A] gave a deep connection between inverse semigroups and finite semigroups by proving his now famous result which states that any finite semigroup S, the idempotents of which commute with each other, is a homomorphic image of a subsemigroup T of some finite inverse semigroup I, in which case we say that S **divides** I. (The converse is of course clearly also true.) This can be also be stated within the context of pseudovariety theory: the pseudovariety of finite semigroups generated by finite inverse semigroups is precisely the pseudovariety of finite semigroups whose idempotents commute.

Previous to this, Birget [B] (or, for recent proofs, see [G] and [BM]) proved that any (finite) semigroup S whose principal left and right ideals form forests under inclusion (such semigroups are called unambiguous) embeds into a (finite) regular semigroup S_{reg} such that S and S_{reg} have the same maximal subgroups. In particular, since it is known that every (finite) semigroup is a homomorphic image of a (finite) unambiguous semigroup (via the Rhodes expansion [T]), it follows that every finite semigroup with trivial subgroups (called aperiodic semigroups) divides a regular aperiodic semigroup. Again from the point of view of pseudovarieties, this says that the pseudovariety of aperiodic semigroups is generated by its regular members.

The purpose of this paper is to study the question arising from the intersection of the classes in the theorems of Ash and Birget: does every finite aperiodic semigroup with commuting idempotents divide a finite aperiodic inverse semigroup? That is, is the pseudovariety **AInv** generated by finite aperiodic inverse semigroups equal to the pseudovariety **AIC** of aperiodic semigroups with commuting idempotents? Surprisingly we show that the answer is no. We go on to show that a pseudovariety **V** generated by inverse semigroups will never contain **AIC** as long as the groups of **V** are compelled to lie within any *proper* pseudovariety of groups.

In section 2 we show that the pseudovariety generated by the finite aperiodic inverse semigroups is strictly contained in **AIC**, the pseudovariety of aperiodic semigroups with commuting idempotents. In section 3 we generalize the method introduced here to prove that the smallest pseudovariety containing **AIC** which is generated by inverse semigroups is the pseudovariety **IC** of all finite semigroups with commuting idempotents.

On the other hand, in section 4, we give a positive result by proving that every semigroup S in **IC** divides a semigroup $\tilde{S}$ where $\tilde{S}$ is a semigroup of partial bijections and the maximal subgroups of $\tilde{S}$ are the same as those of S. In particular, **AIC** is generated by its semigroups of partial bijections. The subtle difference between this result and those in sections 2 and 3 is that here the semigroups $\tilde{S}$ are not necessarily closed under inversion. In the final section we show how our construction can be used to settle an open question concerning semidirect products of $\mathcal{R}$-trivial semigroups and pseudovarieties of groups.

2. The aperiodic case

In this section we construct a finite aperiodic semigroup S with commuting idempotents and prove that it does not divide any finite aperiodic inverse semigroup. Our semigroup S is to be a certain 19-element subsemigroup of I_4, the symmetric inverse semigroup on the set $X = \{1,2,3,4\}$. Let B be the ideal of I_4:

$$B = \{\alpha \in I_4 : \ |\operatorname{dom} \alpha| \leq 1\}.$$

We see that $|B| = 4^2 + 1 = 17$; indeed it is readily checked that B is a combinatorial Brandt semigroup with 4 non-zero idempotents. We then form S by adjoining to B the two elements a' and b' as follows:

$$a' = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & - & - \end{pmatrix}, \quad b' = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & - & - \end{pmatrix}.$$

We see at once that $a'^2 = b'^2 = a'b' = b'a' = 0$, so that $S = B \cup \{a', b'\}$ is a 19-element aperiodic subsemigroup of I_4, and so in particular S has commuting idempotents. We shall make use of the following fact taken from [RW].

Lemma 2.1: *Let ϕ: $T \to S$ be a surjective homomorphism of finite semigroups and let J' be a $\mathcal{J}$-class of S. Then $J'\phi^{-1} = J_1 \cup J_2 \cup \cdots \cup J_k$ is a union of $\mathcal{J}$-classes of T, and if J_i $(1 \leq i \leq k)$ is $\leq_{\mathcal{J}}$-minimal among $J_1, \ldots, J_k$, then $J_i\phi = J'$. Furthermore, if J' is regular, then the index i is uniquely determined, that is J_i is $\leq_{\mathcal{J}}$-minimum among $J_1, \ldots, J_k$, and J_i is itself regular.*

Proposition 2.2: *The finite semigroup S is aperiodic with commuting idempotents but does not divide any finite aperiodic inverse semigroup I.*

Proof: Suppose that there existed a surjective homomorphism ϕ: $T \to S$, where T was a subsemigroup of a finite aperiodic inverse semigroup I.

Let J be the unique minimum (regular) $\mathcal{J}$-class of T such that $J\phi = J'$, the 4×4 $\mathcal{J}$-class of S. Let Z be the ideal of T, $0\phi^{-1}$. Then from the minimality property of J it follows that $C = J \cup Z$ is a subsemigroup of T such that $C\phi = B$. Consider the Rees quotient C/Z. Since Z is a kernel class of the mapping ϕ, it follows that ϕ induces a surjective homomorphism $\tilde{\phi}$: $C/Z \to B$. Hence B is a homomorphic image of the finite aperiodic Brandt semigroup C/Z; however, since such semigroups are congruence-free, it follows that $\tilde{\phi}$: $C/Z \to B$ is an isomorphism. Therefore we can conclude that J is also a 4×4 regular combinatorial $\mathcal{J}$-class of T.

We shall denote by $(i,j)'$ the member of B which maps i onto j $(i,j \in \{1,2,3,4\})$, and denote the unique inverse image in J of $(i,j)'$ under ϕ by (i,j).

Choose and fix members a and b of T such that $a\phi = a'$ and $b\phi = b'$. We complete the demonstration by verifying that the monogenic subsemigroup $A = \langle ba^{-1}\rangle$ of T contains a non-trivial subgroup. In order to do this it is sufficient to show that the right multiplicative action of ba^{-1} on the point $(1,1)$ of T is that of a non-trivial cycle.

We continue the proof of the proposition by analyzing the actions of a and b on the members of J. Suppose that in S, $(i,j)'a' = (k,l)'$. This gives

$$((i,j)a)\phi = (i,j)\phi a\phi = (i,j)'a' = (k,l)' = (k,l)\phi. \tag{1}$$

Now $(i,j)a \leq_{\mathcal{J}} (i,j)$ and, since $((i,j)a)\phi \in J'$, the minimality condition on J ensures that this inequality of $\mathcal{J}$-classes cannot be strict. Hence $(i,j)a \in J$, and since ϕ is one-to-one on J we conclude from (1) that $(i,j)a = (k,l)$. Conversely, if $(i,j)a = (k,l)$ it follows from the fact that ϕ is a homomorphism that $(i,j)'a' = (k,l)'$. The same argument applies equally well to b, or to the reverse products. In conclusion, if we let c stand for either of a or b we have

$$(i,j)c = (k,l) \quad \text{iff } (i,j)'c' = (k,l)' \quad \text{and} \quad c(i,j) = (k,l) \quad \text{iff } c'(i,j)' = (k,l)'.$$

That is to say, a and b act on J as a' and b' act on J'. Moreover, since $(j,i)'$ is inverse to $(i,j)'$ in J', it follows that in the inverse semigroup I, $(i,j)^{-1} = (j,i)$.

To complete the proof it is sufficient to verify that in I,

$$(1,1)ba^{-1} = (1,2) \quad \text{and} \quad (1,2)ba^{-1} = (1,1).$$

Now, since $(1,1)'b' = (1,4)'$ it follows that $(1,1)b = (1,4)$; similarly $a(4,1) = (2,1)$. Hence we obtain

$$(1,1)ba^{-1} = (1,4)a^{-1} = (((1,4)a^{-1})^{-1})^{-1} = (a(4,1))^{-1} = (2,1)^{-1} = (1,2).$$

The following similar calculation completes the proof:

$$(1,2)ba^{-1} = (1,3)a^{-1} = (((1,3)a^{-1})^{-1}))^{-1} = (a(3,1))^{-1} = (1,1)^{-1} = (1,1).$$

3. The general case

Let $b_1, b_2, \ldots, b_k$ be injective mappings on the set $X_n = \{1,2,\ldots,n\}$, and let U be the semigroup generated by the injections b_i, $(1 \leq i \leq k)$. We build the following subsemigroup $S(U)$ of the symmetric inverse semigroup I_{2n}, the base set of which we shall take to be $X_{2n} = \{1,2,\ldots,n,1',2',\ldots,n'\}$. For a subset $D = \{i_1, i_2, \ldots, i_t\}$ of X_n denote by D' the set $\{i_1', i_2', \ldots, i_t'\}$. For each i

$(1 \le i \le k)$ let b_i' be the map with $\operatorname{dom} b_i' = \operatorname{dom} b_i$, and $\operatorname{ran} b_i' = (\operatorname{ran} b_i)'$ which acts as follows:

$$j \cdot b_i' = (j \cdot b_i)'.$$

Similarly, let a' be the map with domain X_n and range X_n' for which $j \cdot a' = j'$.

Finally let $S(U)$ be the semigroup generated by the mappings b_i' $(1 \le i \le k)$, together with a' and B, the combinatorial Brandt semigroup consisting of all the mappings of I_{2n} of rank no more than 1.

For example, the semigroup S of the previous section is a special case of this construction: there $n = 2$ and the elements 3 and 4 correspond to $1'$ and $2'$ respectively; moreover $k = 1$, and the unique injection $b_1 = b$ of $\{1, 2\}$ is the transposition (1 2), and so our U is a two-element group.

As before, $S(U)$ is the disjoint union of B, the combinatorial Brandt semigroup of all mappings in I_{2n} of rank at most 1 (which has $(2n)^2+1$ elements and $2n$ non-zero idempotents), and the set $\{b_1', b_2', \ldots, b_k', a'\}$, as this latter set generates only a zero semigroup, as ranges and domains are disjoint. Thus $|S(U)| = 4n^2 + k + 2$ (in the previous section we saw $n = 2$ and $k = 1$ to yield our 19-element semigroup S). It follows that $S(U)$ is aperiodic with commuting idempotents. We shall follow the argument of Proposition 2.2 to prove the main result, Theorem 3.2. We shall, however, require one basic fact concerning Brandt semigroups.

LEMMA 3.1: *The only congruence ρ on a Brandt semigroup S that is not contained in $\mathcal{H}$ is the universal congruence ω.*

Proof: Suppose that $\rho \not\subseteq \mathcal{H}$. Certainly if $(a, 0) \in \rho$ with $a \neq 0$ then the fact that S is 0-simple gives immediately that $\rho = \omega$. On the other hand, suppose that a and b are not 0, and that $(a, b) \in \rho$ but that $(a, b) \notin \mathcal{H}$. Suppose that $(a, b) \notin \mathcal{L}$. Taking the unique idempotent $e \in E(L_b)$ we obtain $b = be\rho ae = 0$, whence $\rho = \omega$ by the previous argument. The dual argument yields the same conclusion in the alternative case where $(a, b) \notin \mathcal{R}$. ∎

THEOREM 3.2: *If $S(U)$ is a divisor of some finite inverse semigroup I, then U divides I also.*

Proof: Suppose that $\phi\colon T \to S(U)$ is a surjective homomorphism from a semigroup T which is a subsemigroup of some finite inverse semigroup I. Let J' denote the major $\mathcal{J}$-class of B and, again invoking Lemma 2.1, let J be the unique minimum (regular) $\mathcal{J}$-class of T such that $J\phi = J'$. We proceed as in the proof of Proposition 2.2 to conclude that B is a homomorphic image of C/Z, a completely 0-simple inverse semigroup, that is a Brandt semigroup over some group H. That the major $\mathcal{J}$-class J of C/Z is also a $2n \times 2n$ array (and not some strictly larger one) is a consequence of Lemma 3.1.

We denote by $(i,j)'$ the member of B which maps i onto j $(i,j \in \{1,2,\ldots,n,1',2',\ldots,n'\})$. From Lemma 3.1 it follows that $(i,j)'\phi^{-1} \cap J$ is a single $\mathcal{H}$-class $H_{(i,j)}$ contained in J, and not a union of several such $\mathcal{H}$-classes. (We shall often write $H_{i,j}$.) Hence ϕ induces a bijection between the $\mathcal{H}$-classes within J and those of J', which are of course singletons $(i,j)'$. Since ϕ preserves $\mathcal{L}$- and $\mathcal{R}$-classes it follows that J consists of $2n$ $\mathcal{R}$-classes and $2n$ $\mathcal{L}$-classes respectively indexed as follows:

$$R_i = H_{i,1} \cup H_{i,2} \cup \cdots \cup H_{i,2n};$$
$$L_i = H_{1,i} \cup H_{2,i} \cup \cdots \cup H_{2n,i} \quad (1 \leq i \leq 2n).$$

Choose and fix members a and B_i $(1 \leq i \leq k)$ of T such that $a\phi = a'$ and $B_i\phi = b_i'$. Let $c \in \{a, B_1, B_2, \ldots, B_k\}$, and take B_i' to stand for b_i'. After the fashion of the proof of Proposition 2.2, suppose that in $S(U)$ we have $(i,j)'c' = (k,l)'$. Then since $(k,l)' \leq_{\mathcal{R}} (i,j)'$ and J' is regular it follows that $(i,j)'\mathcal{R}(k,l)'$, so that $k = i$. This gives

$$(2) \qquad (H_{i,j}c)\phi = H_{i,j}\phi c\phi = (i,j)'c' = (i,l)' = H_{i,l}\phi.$$

Again, the fact that $(H_{i,j}c)\phi \in J'$ together with the minimality condition on J ensures that $H_{i,j}c \subseteq J$. Now for any member $x \in H_{i,j}$, $xc \leq_{\mathcal{R}} x$, and $x\mathcal{J}xc$, whence it follows that $x\mathcal{R}xc$ as no two distinct $\mathcal{R}$-classes within the regular $\mathcal{J}$-class J are comparable. It follows from Green's Lemma that right multiplication by c defines a bijection of $H_{i,j}$ onto the $\mathcal{H}$-class $H_{i,j}c$, and since the action induced by ϕ on the $\mathcal{H}$-classes of J' is one-to-one, it follows that $H_{i,j}c = H_{i,l}$. Conversely, if $H_{i,j}c = H_{i,l}$, it follows from the fact that ϕ is a homomorphism that $(i,j)'c' = (i,l)'$. Combining this analysis with its dual we see that c acts on the $\mathcal{H}$-classes of J as c' acts on the members of J' in that the actions are both defined or not defined together, and if defined they take the form

$$H_{i,j}c = H_{i,l} \quad \text{iff } (i,j)'c' = (i,l)' \quad \text{and} \quad cH_{i,j} = H_{l,j} \quad \text{iff } c'(i,j)' = (l,j)'.$$

Furthermore, since $(j,i)'$ is inverse to $(i,j)'$ in J', it follows that in the inverse semigroup I, the set of inverses of $H_{i,j}$, which we write as $H_{i,j}^{-1}$, is equal to $H_{j,i}$.

Now let $b \in \{b_1, b_2, \ldots, b_k\}$, and write B for the chosen member of $b'\phi^{-1}$. If

$j \in \operatorname{dom} b$, then $H_{i,j}B = H_{(i,j)b'} = H_{i,(j\cdot b)'}$. Thus we obtain

$$\begin{aligned} H_{i,j}Ba^{-1} &= H_{i,(j\cdot b)'}a^{-1} = (((H_{i,(j\cdot b)'}a^{-1})^{-1})^{-1} \\ &= (aH_{(j\cdot b)',i})^{-1} = (H_{j\cdot b,i})^{-1} = H_{i,j\cdot b}. \end{aligned}$$

It follows that

$$(3) \qquad L_jBa^{-1} = L_{j\cdot b} \quad \text{if } j \in \operatorname{dom} b \quad \text{and} \quad L_jBa^{-1} \cap J = \emptyset \quad \text{otherwise.}$$

We finish the proof by showing that the semigroup U is a homomorphic image of the subsemigroup A of I generated by $\{B_1a^{-1}, B_2a^{-1}, \ldots, B_ka^{-1}\}$. We claim that the mapping whereby $Ba^{-1} \mapsto b$ induces a homomorphism of A onto U. To justify this we are required to check that if two products $x = B_{i_1}a^{-1}B_{i_2}a^{-1}\cdots B_{i_p}a^{-1}$ and $y = B_{j_1}a^{-1}B_{j_2}a^{-1}\cdots B_{j_q}a^{-1}$ represent equal members of A, then their respective images $\tilde{x}$ and $\tilde{y}$ in U are also equal.

To this end, take any $j \in \{1, 2, \ldots, n\}$, and suppose that $j \cdot \tilde{x}$ is defined. Then we obtain from p-fold and q-fold use of (3)

$$L_{j\cdot\tilde{x}} = L_jx = L_jy = L_{j\cdot\tilde{y}},$$

whence $j \cdot \tilde{x} = j \cdot \tilde{y}$ for all $j \in \operatorname{dom} x$; by the same argument, $j \cdot \tilde{y} = j \cdot \tilde{x}$ for all $j \in \operatorname{dom} y$, which yields the required conclusion $\tilde{x} = \tilde{y}$. ∎

COROLLARY 3.3: *Let* $\mathbf{V}$ *be a pseudovariety of semigroups generated by a collection of inverse semigroups. Then*

$$\mathbf{A} \cap \mathbf{IC} \subseteq \mathbf{V} \Rightarrow \mathbf{V} = \mathbf{IC}.$$

Proof: Let U be any semigroup of one-to-one mappings on some finite set. Construct the finite semigroup $S(U)$ as above. Since $S(U)$ is aperiodic with commuting idempotents, $S(U) \in \mathbf{V}$. Since $\mathbf{V}$ is generated by inverse semigroups it also follows that $S(U)$ divides some finite inverse semigroup I such that $I \in \mathbf{V}$. Then by Theorem 3.2, U divides I as well, whence $U \in \mathbf{V}$. Therefore $\mathbf{V}$ contains all such semigroups U, whence, by Ash's Theorem, $\mathbf{IC} \subseteq \mathbf{V}$; the reverse inclusion is certainly true. ∎

COROLLARY 3.4: *Let* $\mathbf{G}$ *be a proper pseudovariety of groups, let* $\mathbf{V}(\mathbf{G})$ *be the pseudovariety of all semigroups the subgroups of which lie in* $\mathbf{G}$. *Then there exists a finite aperiodic semigroup with commuting idempotents that does not divide any inverse semigroup in* $\mathbf{V}(\mathbf{G})$.

Proof: Take G to be a finite group not in $\mathbf{G}$. Then $S(G) \in \mathbf{A} \cap \mathbf{IC}$. However, if $S(G)$ divides a finite inverse semigroup I, then so does G, and so any such I is not a member of $\mathbf{V}(\mathbf{G})$. ∎

Remark: Theorem 3.2 in fact allows us to replace the pseudovariety $\mathbf{A}$ by a smaller pseudovariety and thereby gain a stronger statement than Corollary 3.3. The pseudovariety $\mathcal{V}$ in question is contained in that generated by all ideal extensions of aperiodic Brandt semigroups by zero semigroups. Now the pseudovariety generated by the class of aperiodic Brandt semigroups is known to be given by the following identities [Tr],

$$[x^2 = x^3, xyx = xyxyx, xyxzx = xzxyx, x^2y^2 = y^2x^2]. \tag{4}$$

The generators of $\mathbf{V}$ are our semigroups $S(U)$ which have the additional property that the complement of the major $\mathcal{J}$-class forms a zero subsemigroup. It follows from this observation together with the fact that $S(U)^2$ satisfies the equations (4), that $\mathbf{V}$ is contained in the pseudovariety $\mathbf{W}$ given by the equations $x^2 = x^3$ together with

$$xyzwxy = xyzwxyzwxy, \quad xyzwxytrzwxy = xytrxyzwxy, \quad x^2y^2 = y^2x^2.$$

Therefore we may replace $\mathbf{A}$ by the pseudovariety $\mathbf{W}$ in the statement of Corollary 3.3.

4. A division theorem

We can consider the results of the previous two sections as giving restrictions on the divisors of finite inverse semigroups when we restrict the class of maximal subgroups of such semigroups. In this section we give a positive result by proving that every finite semigroup S with commuting idempotents divides a semigroup of partial bijections $\tilde{S}$ such that S and $\tilde{S}$ share the same maximal subgroups. We do this by showing that the semigroup constructed in [BMR] satisfies these properties. This semigroup was created to help give a more algebraic proof of Ash's Theorem [A] on commuting idempotents and to give some generalizations.

Throughout, S will denote a finite semigroup whose idempotents commute. One well-known and easily verified property of such a semigroup is that the set $\text{Reg}(S)$ of regular elements of S is an inverse semigroup.

We recall the definition and most important properties of $\tilde{S}$. Let S be a finite semigroup with commuting idempotents. The semigroup $\tilde{S}$ is defined to be the semigroup with generators the set S and a set of relations to be described below. We need to distinguish between products of elements of S considered as a semigroup and elements of the free semigroup on the set S. If $s_1, \ldots, s_n$ is a sequence of elements of S, then $w = (s_1, \ldots, s_n)$ denotes the word of length n in the free semigroup on S and $s_1 \cdots s_n$ denotes the product of these elements in S. In particular, $(s_1 \cdots s_n)$ denotes the string of length one containing the element $s_1 \cdots s_n$.

Now we describe the relations of $\tilde{S}$. It is the collection of all relations of the form $(s_1, \ldots, s_n) = (s_1 \cdots s_n)$ such that $n > 1$ and the product $s_1 \cdots s_n$ is a regular element of the semigroup S. That is, we replace a string w of elements of S by the string of length one equal to the product of the elements in w *if and only if* this product is a regular element of the semigroup S. We also consider the rewrite system [Sims] associated with this relation by orienting rules so as to replace strings of length greater than 1 by the string of length one. It is clear then that this length reducing restriction shows that the rewrite system is Noetherian. That is, there are no infinite descending chains of rewrite substitutions.

This is an infinite collection of relations, but nonetheless, the semigroup $\tilde{S}$ is finite and has some remarkable properties as outlined in the following theorem. The proofs of the theorems can be found in Sections 4 and 5 of [BMR]. The idea for the construction and the proofs of some parts of the following were motivated by the work of Ash. In particular, they use Ramsey's Theorem in a non-trivial way.

THEOREM 4.1:

1. *The canonical map $S^+ \to S$ factors through the canonical map $S^+ \to \tilde{S}$. Thus there is an induced morphism $f\colon \tilde{S} \to S$.*
2. *The rewrite system is confluent and Noetherian. That is, every element of $\tilde{S}$ is represented by a unique word that is not reducible by the rewrite rules.*
3. *For each regular element $r \in S$, rf^{-1} consists of exactly one element of $\tilde{S}$ whose canonical representative is the string of length one (r).*
4. *$\tilde{S}$ is a finite semigroup whose idempotents commute and* $\mathrm{Reg}(\tilde{S})$ *is isomorphic to* $\mathrm{Reg}(S)$. *In particular, every maximal subgroup G of $\tilde{S}$ is isomorphic to the maximal subgroup Gf of S.*
5. *$\tilde{S}$ has a faithful representation by partial bijections on a finite set.*

COROLLARY 4.2: *Every finite semigroup S whose idempotents commute is the homomorphic image of a semigroup $\tilde{S}$ such that*

1. *S and $\tilde{S}$ have the same maximal subgroups,*
2. *$\tilde{S}$ is faithfully represented by partial bijections on a finite set.*

Now let $\mathbf{H}$ denote a pseudovariety of finite groups and let $\mathbf{V}(\mathbf{H})$ denote the pseudovariety of finite semigroups all of whose subgroups belong to $\mathbf{H}$. Let $\mathbf{ICH} = \mathbf{IC} \cap \mathbf{V}(\mathbf{H})$ be the pseudovariety of all finite semigroups whose idempotents commute and subgroups belong to $\mathbf{H}$. In particular, if $\mathbf{T}$ is the trivial pseudovariety and $\mathbf{G}$ is the pseudovariety of all finite groups, then $\mathbf{ICT}$ is the pseudovariety of all aperiodic idempotent commuting semigroups and $\mathbf{ICG} = \mathbf{IC}$, the pseudovariety of all finite semigroups whose idempotents commute. The following theorem follows immediately from the previous corollary.

THEOREM 4.3: *Let $\mathbf{H}$ be a pseudovariety of groups. Every $S \in \mathbf{ICH}$ is a homomorphic image of a semigroup $\tilde{S}$ of partial bijections that also belongs to $\mathbf{ICH}$. In particular, $\mathbf{ICH}$ is generated by its semigroups of partial bijections.*

5. Answering a question of Almeida

Let $\mathbf{ER}$ denote the pseudovariety of all finite monoids whose idempotents generate an $\mathcal{R}$-trivial submonoid. An important theorem of Stiffler shows that this pseudovariety decomposes as the product $\mathbf{R} * \mathbf{G}$ of the pseudovariety of $\mathcal{R}$-trivial monoids and the pseudovariety of finite groups. See [Al], Theorem 10.10.8. Almeida [Al, Problem 10.10.19] asked if this decomposition can be relativized to some other pseudovariety $\mathbf{H}$ of groups. That is, he wanted to know if the equation

$$\mathbf{ER} \cap \mathbf{V}(\mathbf{H}) = \mathbf{R} * \mathbf{H}$$

holds. In this section we prove that this equation never holds for any proper pseudovariety of groups. Clearly the above equation is false if $\mathbf{H}$ is the trivial variety, so let us assume the $\mathbf{H}$ is non-trivial and proper. Let S be any semigroup in $\mathbf{R} * \mathbf{H}$. Thus there is an $\mathcal{R}$-trivial semigroup T, a group $G \in \mathbf{H}$ that acts on T such that S divides the semidirect product $T * G$. By standard arguments [P], this induces a relational morphism $\tau\colon S \to G$. This means that $\mathrm{graph}(\tau) = \{(s,g) | g \in s\tau\}$ is a subsemigroup of $S \times G$ that projects onto S and that $\mathrm{Ker}(\tau) = 1\tau^{-1}$ is an $\mathcal{R}$-trivial subsemigroup of S. The following lemma is a well known property of kernels onto groups.

LEMMA 5.1: *Let $r \in \mathrm{Ker}(\tau)$ and let s be any element of S such that $srs = s$. Then $s \in \mathrm{Ker}(\tau)$.*

Proof: Since $r \in \mathrm{Ker}(\tau)$, $(r,1) \in \mathrm{graph}(\tau)$. Let g be any element of G such that $(s,g) \in \mathrm{graph}(\tau)$. (Such an element exists by the definition of relational morphism.) Then $(rs,g) \in \mathrm{graph}(\tau)$. But rs is an idempotent and thus $(rs,g^k) \in$

graph(τ) for all $k > 0$. In particular, there is a k such that $g^k = g^{-1}$ and thus $(rs, g^{-1}) \in$ graph(τ). Therefore, $(s,g)(rs,g^{-1}) = (srs, 1) = (s, 1)$. Thus $s \in \mathrm{Ker}(\tau)$. ∎

COROLLARY 5.2: *Let $S \in \mathbf{ER}$ and let $\tau: S \to G$ be a relational morphism onto a group such that* $\mathrm{Ker}(\tau)$ *is $\mathcal{R}$-trivial. If $r \in \mathrm{Ker}(\tau)$ is a regular element of S, then $r = r^2$.*

Proof: By Lemma 5.1, r is in fact a regular element of $\mathrm{Ker}(\tau)$. Assume that r is not an idempotent. Since $\mathrm{Ker}(\tau)$ is $\mathcal{R}$-trivial, r cannot belong to a group in S. Let s be an inverse of r in $\mathrm{Ker}(\tau)$. Then since $S \in \mathbf{ER}$, it follows that $\{r, rs, sr, s\}$ is a set such that $r\mathcal{R}rs\mathcal{L}s\mathcal{R}sr\mathcal{L}r$ in $\mathrm{Ker}(\tau)$. Since r is assumed not to be an idempotent, r is not equal to rs and this contradicts the fact that $\mathrm{Ker}(\tau)$ is $\mathcal{R}$-trivial. ∎

THEOREM 5.3: *Let $\mathbf{H}$ be a proper non-trivial pseudovariety of finite groups and let G be a finite group not in $\mathbf{H}$. Then the aperiodic semigroup $S(G)$ (as constructed in Section 3) belongs to $\mathbf{ER}$ but not to $\mathbf{R} * \mathbf{H}$.*

Proof: Let G be a group that is not in $\mathbf{H}$. Clearly $S(G)$ is an aperiodic semigroup in $\mathbf{ER}$. Assume that the semigroup $S(G)$ is in $\mathbf{R} * \mathbf{H}$. Then there is a group K in $\mathbf{H}$ and a relational morphism $\tau: S(G) \to K$ such that $\mathrm{Ker}(\tau)$ is $\mathcal{R}$-trivial. By the corollary, the regular elements of $\mathrm{Ker}(\tau)$ are just the idempotents of $S(G)$. Now $S(G)$ has the structure of a unique 0-minimal ideal which is the union of a combinatorial Brandt semigroup and the set $\{a_i | i = 1, \ldots, k\}$, such that the product of any pair is the zero of $S(G)$. Let L be any nontrivial element of $\mathbf{H}$ and let g be a non-identity element of L. Consider the relational morphism $\tau_1: S(G) \to L$ such that $(a_i)\tau_1 = g$ and $s\tau_1 = L$ for all s in the 0-minimal ideal. (It is a relational morphism since the product of the a_i are always 0.) Now none of the a_i are in $\mathrm{Ker}(\tau_1)$. Consider the product relational morphism $\tau \times \tau_1: S(G) \to K \times L$ defined by $(s)(\tau \times \tau_1) = s\tau \times s\tau_1$. Then $\mathrm{Ker}(\tau \times \tau_1) = \mathrm{Ker}(\tau) \cap \mathrm{Ker}(\tau_1)$ is contained in the 0-minimal ideal. By the corollary, $\mathrm{Ker}(\tau \times \tau_1)$ consists only of the idempotents of $S(G)$, which form a semilattice and thus $S(G)$ is in the Malcev product of the pseudovariety $\mathbf{Sl}$ of all semilattices and $\mathbf{V}(\mathbf{H})$ which is known to be equal to $\mathbf{Sl} * \mathbf{V}(\mathbf{H})$. This is a pseudovariety generated by inverse semigroups whose subgroups lie in $\mathbf{H}$, contradicting Theorem 3.2. ∎

The inequality represented by Theorem 5.3 can indeed be generalized to a wider class. Let $\mathbf{V}$ be a pseudovariety such that $\mathbf{Sl} \subseteq \mathbf{V} \subseteq \mathbf{DA}$ where $\mathbf{DA}$ is the pseudovariety of all finite monoids whose regular $\mathcal{D}$-classes are rectangular bands. Let $\mathbf{EV}$ then stand for the the pseudovariety of all finite monoids whose idempotents generate a monoid in $\mathbf{V}$. It is easy to see that the Malcev product [P] $\mathbf{V}m\mathbf{H}$ is contained in $\mathbf{EV} \cap \mathbf{V}(\mathbf{H})$, when $\mathbf{H}$ is any pseudovariety of groups. We now use the fact that $\mathbf{V}m\mathbf{H}$ is the pseudovariety of all finite semigroups S that have a relational morphism $\tau\colon S \to G$, where $G \in \mathbf{H}$ and $\operatorname{Ker}\tau = 1\tau^{-1} \in \mathbf{V}$. We first have the analogue of Corollary 5.2.

LEMMA 5.4: *Let $S \in \mathbf{EV}$. Let $\tau\colon S \to G$ be a relational morphism onto a group such that* $\operatorname{Ker}(\tau) \in \mathbf{V}$. *Then every regular element r of S is an idempotent.*

Proof: The argument is the same as that of Corollary 5.2 but in this instance the contradiction is furnished from $r\mathcal{R}rs\mathcal{L}s\mathcal{R}sr\mathcal{L}r$ by noting that the assumption that $r \neq rs$ would give us a semigroup *not* in $\mathbf{DA}$. ∎

Thus we have the following generalization of Theorem 5.3.

THEOREM 5.5: *Let $\mathbf{Sl} \subseteq \mathbf{V} \subseteq \mathbf{DA}$. Let $\mathbf{H}$ be a proper pseudovariety of groups and let G be a finite group not in $\mathbf{H}$. Then $S(G)$ lies in $\mathbf{EV}$ but not in $\mathbf{V}m\mathbf{H}$.*

Remark: It is known that the equality $\mathbf{V}m\mathbf{G} = \mathbf{EV}$ holds for the pseudovariety of all finite groups for any $\mathbf{V}$ such that $\mathbf{Sl} \subseteq \mathbf{V} \subseteq \mathbf{DA}$.

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